Math 246A Lecture 21 Notes

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November 14, 2018

1 Principal Value and the Dirichlet Problem

1.1 Principal value

Definition 1.1. Let $f(z)$ be meromorphic. The **principal value** of f is

$$PV \int_{-\infty}^{\infty} f(x)\, dx = \lim_{\substack{A,B\to\infty \\ \varepsilon\to 0}} \int_{[-A,B]\setminus \bigcup_{j=1}^{n}\{|x-a_j|<\varepsilon\}} f(x)\, dx.$$

Theorem 1.1. *Let $f(z)$ be meromorphic in $U \supseteq \{z : \mathrm{Im}(z) > 0\}$, and assume that $|f(z)| \leq K/|z|$ as $z \to \infty$. f has poles $a_1, \dots, a_n$ on $\mathbb{R}$, all simple poles. Then the principal value is*

$$PV \int_{-\infty}^{\infty} f(x)e^{i\lambda x}\, dx = 2\pi i \sum_{\substack{\mathrm{Im}(a)>0 \\ a \text{ a pole}}} \mathrm{Res}(e^{i\lambda z} f(z), a) + \pi i \sum_{j=1}^{n} \mathrm{Res}(e^{i\lambda z} f(z), a_j)$$

Proof. Create a contour with a rectangle in the upper half plane, which has little indents to avoid the poles. Here, γ_1 is the bottom of the rectangle (with the little circular indents). By the residue theorem,

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} = 2\pi i \sum \mathrm{Res}(e^{i\lambda z} f(z), a).$$

$$\left| \int_{\gamma_2} \right| eq \frac{K}{B} \int_0^B e^{-\lambda y}\, dy \to 0,$$

and $\left| \int_{\gamma_4} \right| \to 0$ in the same way. Also,

$$\left| \int_{\gamma_3} \right| \leq \frac{(A+B)K}{M} \to 0$$

as $M \to \infty$.

$$\oint_{|z-a_j=\varepsilon \operatorname{Im}(z)>0} = -\int_0^\pi f(z)e^{i\lambda z}\varepsilon i e^{it}\,dt,$$

where $z = a_j + \varepsilon e^{it}$. Since

$$f(z)e^{i\lambda z} = \frac{\operatorname{Res}(f(z)e^{i\lambda z}, a_j)}{\varepsilon e^{it}} + \underbrace{\sum_{k=0}^{\infty} A_k(z-a_j)^k}_{\to 0},$$

we get the result. $\square$

Example 1.1. Let $0 < \beta < 1$. Then let's solve

$$\int_0^\infty \frac{x^\beta}{1+x^2}\,dx$$

We use a "keyhole contour," consisting of a small circle around 0 connected to a large circle of radius R. Let γ_1 be the contour along the real axis going from the small circle to the large circle, let γ_2 be the large circle, let γ_3 be the reverse real axis contour, and let γ_4 be the small circle. It's important to notice that γ_1 and γ_3 don't cancel because $z = |z|e^{i\arg(z)}$, and $\arg(z) = 0$ on γ_1, and $\arg(z) = 2\pi$ on γ_2.

$$\int_{\gamma_1} = \int \varepsilon^R \frac{x^\beta}{1+x^2},$$

$$\int_{\gamma_3} = -\int_\varepsilon^R \frac{x^\beta e^{(2\pi i)\beta}}{1+(xe^{2\pi i})^2}\,dx.$$

So we get

$$\gamma_1 + \gamma_3 = (1 - e^{2\pi i\beta})\int_\varepsilon^R \frac{x^\beta}{1+x^2}\,dx.$$

$$\int_{\gamma_4} \xrightarrow{\varepsilon\to 0} 0,$$

and

$$\left|\int_{\gamma_2}\right| \le \frac{R^\beta}{R^2-1} 2\ piR \xrightarrow{R\to\infty} 0.$$

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} = (2\pi i)\left[\operatorname{Res}(z^\beta/(1+z^2), i) + \operatorname{Res}(z^\beta/(1+z^2), -i)\right] = \pi(e^{i\beta\pi/2} - e^{i\beta 3\pi/2}).$$

So we get that

$$I = \frac{\pi(e^{i\beta\pi/2} - e^{i\beta 3\pi/2})}{1-e^{2\pi i\beta}} = \frac{\pi(e^{i\beta\pi/2} - e^{i\beta 3\pi/2})}{2\cos(\beta\pi/2)}.$$

1.2 The Dirichlet problem

Let u be harmonic. Recall that

1. A domain Ω is simply connected iff there exists some $f \in H(\Omega)$ such that $u = \mathrm{Re}(f)$.
2. Mean value property:
$$\overline{B(z_0, R)} \subseteq \Omega \implies u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{it})\, dt.$$
3. $u(z_0) = \sup_\Omega u(z) \implies u$ is constant on the domain Ω.

The third property follows from the second by a connectedness argument.

Theorem 1.2. *Let $u \in C(\overline{\mathbb{D}})$, and U is real and harmonic on $\mathbb{D}$. Then*

1. *For all $z \in \mathbb{D}$,*
$$u(z) = \frac{1}{2\pi} \int u(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2}\, d\theta$$
2. *If $f \in C(\partial\mathbb{D})$ and $z \in \mathbb{D}$, then the function*
$$v(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} f(e^{it}) \frac{1 - |z|^2}{|e^{-it} - z|^2}\, dt$$
is harmonic.
3. *The function*
$$u(z) = \begin{cases} f(e^{it}) & z = e^{it} \in \partial\mathbb{D} \\ v(z) & z \in \mathbb{D} \end{cases}$$
is continuous on $\overline{\mathbb{D}}$ if $f \in C(\partial\mathbb{D})$.

This solves the **Dirichlet problem** on $\mathbb{D}$. The function
$$\frac{1 - |z|^2}{|e^{it} - z|^2}$$
is called the **Poisson kernel**.

Corollary 1.1. *If a function satisfies the mean value property for all $0 < r < R$ (for some fixed radius R), it is harmonic.*

We will prove this next time.